Exam Symmetry in Physics

Date	June 26, 2024
Time	08:30 - 10:30
Lecturer	D. Boer

- Write your name and student number on every separate sheet of paper
- Raise your hand for more paper
- You are not allowed to use the lecture notes, nor other notes or books
- Use of a calculator is allowed
- The weights of the three exercises and subquestions are given below
- Illegible answers will be not be graded
- Good luck!

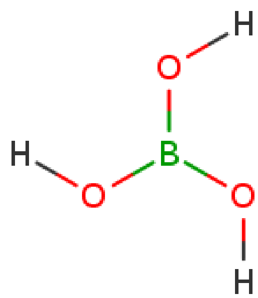
Weighting

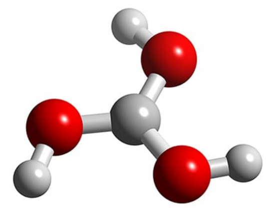
1a)	12	2a)	12	3a)	6
1b)	12	2b)	12	3b)	6
1c)	12	2c)	12	3c)	6

$$\text{Result } = \frac{\sum \text{points}}{10} + 1$$

Exercise 1

Consider the boric acid molecule depicted in the figures below:





All atoms are considered to lie in a plane, but the molecule is a 3-dimensional object.

(a) Identify all symmetry transformations (rotations and reflections) that leave this molecule invariant and call the group that they form G_{BA}. Show that G_{BA} is an Abelian group of order 6.

(b) Construct the character table of G_{BA} and explain how the entries are obtained.

(c) Determine the characters of the three-dimensional vector representations D^V of G_{BA} and use them to determine whether the molecule can have a permanent electric dipole moment or not.

Exercise 2

Consider the symmetric group S_3 consisting of the permutations of three objects and view the three basis vectors of $\mathbb{R}^3$ as the three objects that are permuted. This leads to the three-dimensional representation D^L of S_3, which for the cycles $c = (123)$ and $b = (12)$ is given by:

$$D^L(c) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad D^L(b) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Show that D^L indeed forms a representation of $S_3 \cong D_3$ by using the defining properties of the group (its presentation).

(b) Show using Schur's lemma that D^L is reducible and specify an invariant subspace.

(c) Decompose D^L and $D^L \otimes D^L$ into irreps of S_3 using the character table of S_3:

	(e)	(c)	(b)
$D^{(1)}$	1	1	1
$D^{(2)}$	1	1	-1
$D^{(3)}$	2	-1	0

Exercise 3

Consider the quadrupole (moment) tensor (i, j can take the values 1, 2, and 3):

$$Q_{ij} = \int \rho(r) \left[3r_i r_j - r^2 \delta_{ij}\right] d^3\boldsymbol{r},$$

where $\rho(r)$ denotes a continuous charge density that only depends on the length $r = |\boldsymbol{r}|$ of the position vector $\boldsymbol{r}$.

(a) Determine the number of independent components of the quadrupole tensor and explain under which irreducible representation of $SO(3)$ (the group of rotations in three dimensions) it transforms. The irrep may be specified by its dimension $d = 2l + 1$.

(b) Explain whether or not a quadrupole tensor can be invariant under $SO(3)$.

(c) Determine how the quadrupole tensor behaves under reflection in the origin (space inversion).